and the optimum value of $\lambda_{\max}[P_\tau(0)]$ was

$$\lambda_{\max}[P_\tau(0)] = -0.40844 < 0$$

which indicates that this system has no robustly unobservable states.

For the optimal value of τ given above, a plot of $\lambda_{\max}[P_{\tau^*}(t)]$ as a function of t is shown in Fig. 6.

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Stable Flocking of Multiple Inertial Agents on Balanced Graphs

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***Abstract*—In this note, we consider the flocking of multiple agents which have significant inertias and evolve on a balanced information graph. Here, by flocking, we mean that all the agents move with a common velocity while keeping a certain desired internal group shape. We first show that flocking algorithms that neglect agents' inertial effect can cause unstable group behavior. To incorporate this inertial effect, we use the passive decomposition, which decomposes the closed-loop group dynamics into two decoupled systems: a shape system representing the internal group shape and a locked system describing the motion of the center-of-mass. Then, analyzing the locked and shape systems separately with the help of graph theory, we propose a provably stable flocking control law, which ensures that the internal group shape is exponentially stabilized to a desired one, while all the agents' velocities converge to the centroid velocity that is also shown to be time-invariant. This result still holds for slow-switching balanced information graphs. Simulation is performed to validate the theory.**

***Index Terms*—Decomposition, distributed coordination, inertial effect, information graph, multiagent flocking.**

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I. INTRODUCTION

The multiagent distributed coordination problem has received much attention from many researchers (see [1] for a collection of such efforts). One of the key research questions is how to design a local action of possibly simple agents (with limited computing power and sensing capability) such that, collectively, a certain desired pattern/behavior can emerge. Once we have an answer to this question, we will be able not only to realize many powerful engineering applications (e.g., mobile sensor networks and distributed robotic surveillance/rescue [2]–[4]), but also to understand many fascinating phenomenon in nature (e.g., schooling of fishes [5] and human collective behavior [6]).

Compared to conventional control problems, the unique challenge of the multiagent distributed coordination is how to analyze the information topology among the agents, i.e., which agents sense/are sensed by which agents. This is important, because it determines how the local action propagates throughout the group. At the same time, this requires a different tool and perspective, since now, very differently from the conventional problems, this information topology is, in general, only partially connected (especially when the number of agents is large) and/or limited (e.g., due to interagent communication mechanism [7]).

To analyze this information topology, graph theory has been used in many works (e.g., [8]–[13]), where the information topology is represented by its information graph. Among them, the majority assumes that the evolution of each agent can be described well enough by its kinematics, i.e., single integrator dynamics (e.g., [8]–[11]). However, in many important applications (e.g., robots, spacecraft), it is not generally possible to directly control the velocity, as most of the actuators (e.g., torque motor and gas jet) can affect only the acceleration through the agents' inertias. Moreover, as shown in this note, for a certain (directed) information topology, this agents' inertial effect can even cause unstable group behaviors. Therefore, it is necessary to incorporate this inertial effect (i.e., double integrator dynamics) into the multiagent distributed coordination control design.

In this note, we present a novel distributed coordination framework for multiple agents with significant inertial effect and evolving on a balanced information graph [8] (i.e., in-degree = out-degree for all agents). This class of balanced graphs includes two practically important classes of information topology: undirected graphs (i.e., interagent communication is two-way [9], [12]) and cyclic graphs [14]. We consider the particular problem of flocking [12]: For n agents, we want $\dot{x}_i(t) - \dot{x}_j(t) \to 0$ and $x_i(t) - x_j(t) \to a_{ij}$ for all $i, j \in \{1, 2, \ldots, n\}$, where $x_i(t) \in \Re$ is the position of the agent i and $a_{ij} \in \Re$ is an offset between the agents i and j to make a certain desired group shape. Here, we assume that this offset a_{ij} is constant and also *compatible* in the sense that $a_{ik} + a_{kj} = a_{ij}$ and $a_{ii} = 0$ (i.e., $a_{ij} = -a_{ji}$) for any

$i, j, k \in \{1, 2, \ldots, n\}$. For simplicity, this note mainly deals with the scalar flocking problem (i.e., $x_i \in \Re$). The obtained results, however, are easily extendable for the multidimensional flocking problem as well (i.e., $\mathbf{x}_i \in \Re^m$). See Section V.

To deal with the agents' inertial effect, we use the passive decomposition [15]–[18], which decomposes the closed-loop group dynamics into two decoupled systems: a shape system representing the internal group shape and a locked system describing the dynamics of the centroid (i.e. center-of-mass). Then, by analyzing the locked and shape systems separately with some results of graph theory, we propose a provably stable flocking control law, which ensures that the internal group shape is exponentially stabilized to a desired one, while all the agents' velocities converge to the centroid velocity that is also shown to be time-invariant. Using the dwell-time concept [19], we can also show that this result still holds for slow-switching balanced information graphs.

To our knowledge, in other distributed coordination schemes, the agents' inertial effect has been considered only for the cases where either the information graph is undirected so that the closed-loop group dynamics becomes usual mass–spring–damper dynamics (e.g., [12], [17], [18], [20]), or all the agents' dynamics are identical so that the effects of the agents' dynamics and the information topology can be separated from each other by using Kronecker algebra [13]. In contrast, our proposed framework is applicable even when the agents' inertias are all different and the information graph is a general balanced graph. In this sense, this note may be thought of as an extension of the results in [8] (i.e., coordination of kinematic agents on a balanced graph) and [12] (i.e., flocking of inertial agents on an undirected graph).

In this note, we mainly confine our attention to the balanced information graphs. This is in part because this class of graphs has a fundamental impact on the flocking behavior of inertial agents as revealed here (see Section IV). We also think that this class of graphs will still be useful in many practical applications, since it includes very frequently used undirected and cyclic graphs. Also, although we could extend the results to the slow-switching graphs, the issue of switching is by no means the main concern of this note. Instead, this note mainly aims to provide a framework to systematically analyze the agents' inertial effect on the flocking behavior when coupled with information topology. Furthermore, among the numerous approaches currently available for this switching issue (e.g., single integrator agents [8], [9], [11] and inertial agents on undirected graphs [12]), to our best knowledge, none of them is directly applicable to our setting. How to extend the results presented here to general nonbalanced information graphs and to more aggressive (or state dependent) switchings are topics for future research.

The rest of this note is organized as follows. Section II provides some preliminary graph theory and an example highlighting the importance of the agents' inertial effect. Then, in Section III, using the passive decomposition, the closed-loop group dynamics is decomposed into the locked and shape systems. In Section IV, the novel flocking algorithm for multiple inertial agents on a balanced graph is presented and its properties are detailed, and, in Section V, its extension to the multidimensional flocking problem is presented. In Section VI, simulations are performed, and Section VII contains some concluding remarks. A conference version of this note has been presented in [26].

II. INFORMATION GRAPH AND MOTIVATING EXAMPLE

Consider the n agents. Then, the information topology among them can be represented by their (weighted and directed) information graph $G := \{\mathcal{V}, \mathcal{E}, \mathcal{W}\}$, where $\mathcal{V} := \{v_1, \ldots, v_n\}$ is the set of nodes (i.e., agents), $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of edges (i.e., ordered pairs of the nodes), and $\mathcal{W} : \mathcal{E} \to \Re^+$ is a map assigning a (positive) weight to each edge such that (s.t.) if $e_{ij} := (v_i, v_j) \in \mathcal{E}$, $\mathcal{W}(e_{ij}) = w_{ij}$, $w_{ij} > 0$.

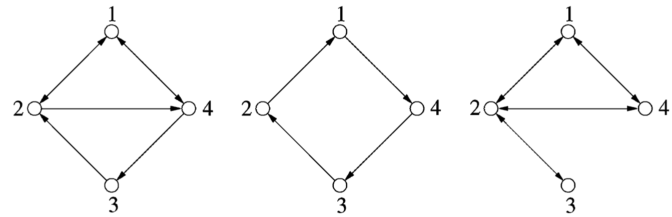

Fig. 1. Some examples of strongly connected balanced information graphs, including cyclic (second) and undirected (third) graphs.

See Fig. 1 for some examples. For notational convenience, we exclude the self-joining edges from $\mathcal{E}$, i.e., $e_{ii} \notin \mathcal{E}$, $\forall i \in \{1, \ldots, n\}$. Here, $e_{ij} = (v_i, v_j) \in \mathcal{E}$ (i.e., v_i and v_j are the head and tail of the edge e_{ij}, respectively) would imply that the information flows from v_j to v_i. This would happen, if the agent i tried to follow the state of the agent j. The weighting w_{ij} can be useful if each information flow has nonuniform reliability [e.g., different signal-to-noise ratio (SNR)]. For more details on the graph theory, refer to [8] and references therein.

Now, following [8] and [10], let us consider the kinematics-based flocking model. Then, the closed-loop kinematics of the agent i can be given by

$$\dot{x}_i = \sum_{j \in \mathcal{N}_i} -k w_{ij}(x_i - x_j) \tag{1}$$

where x_i is the position of the agent i, $w_{ij} > 0$ is the weight assigned to e_{ij}, $k > 0$ is the control gain, and $\mathcal{N}_i$ is the set of the information neighbors of the agent i defined by

$$\mathcal{N}_i := \{j | e_{ij} = (v_i, v_j) \in \mathcal{E}\} \tag{2}$$

i.e., the set of all the tails of the agent i. Here, for simplicity, we assume that the desired group shape is given by $x_1 = x_2 = \ldots = x_n$, although the one with compatible constant offsets a_{ij} can also be easily incorporated.[1]

Then, by stacking up the individual kinematics (1), the closed-loop group kinematics can be written as

$$\dot{x} = -kLx \tag{3}$$

where $x = [x_1, x_2, \ldots, x_n]^T \in \Re^n$, and the matrix $L \in \Re^{n \times n}$ is the Laplacian matrix of the information graph G defined s.t.

$$L_{ij} = \begin{cases} -w_{ij}, & \text{if } i \neq j \text{ and } (v_i, v_j) \in \mathcal{E} \\ 0, & \text{if } i \neq j \text{ and } (v_i, v_j) \notin \mathcal{E} \\ \sum_{k \in \mathcal{N}_i} w_{ik}, & \text{if } i = j \end{cases} \tag{4}$$

where L_{ij} is the ijth component of L.

For node i, we define the in-degree $\mathrm{in}_i(G)$ and the out-degree $\mathrm{out}_i(G)$ s.t. $\mathrm{in}_i(G) := L_{ii}$ $[= -\sum_{j=1, j \neq i}^{n} L_{ij}$ from (4)] and $\mathrm{out}_i(G) := -\sum_{j=1, j \neq i}^{n} L_{ji}$. Note that, if the graph is nonweighted (i.e., $w_{ij} = 1$), $\mathrm{in}_i(G)$ and $\mathrm{out}_i(G)$ are the numbers of incoming and outgoing edges of v_i, respectively.

This graph Laplacian matrix L has the following properties [8], [13]: 1) its eigenvalues have nonnegative real part (from Geršgorin's disk theorem [21]); 2) $u := [1, 1, \ldots, 1]^T \in \Re^n$ is an eigenvector with zero eigenvalue, i.e., $Lu = 0$, since $\sum_{j=1}^{n} L_{ij} = 0$ from (4); and 3)

[1]Define $y_i := x_i + a_{1i}$. Note that, from $a_{ii} = 0$, $y_1 = x_1$. Then, since a_{ij} is constant and compatible, we can incorporate a_{ij} into (1) s.t. $\dot{y}_i = \sum_{j \in \mathcal{N}_i} -k w_{ij}(x_i - x_j - a_{ij}) = \sum_{j \in \mathcal{N}_i} -k w_{ij}(x_i - x_j - a_{i1} - a_{1j}) = \sum_{j \in \mathcal{N}_i} -k w_{ij}(y_i - y_j)$, where the flocking will be achieved if $\dot{y}_i \to \dot{y}_j$ and $y_1 \to y_2$. This is in the same form as (1). Similarly, (5) can also incorporate the offsets a_{ij}. Note that, here, each agent i needs to know a_{ij} for $j \in \mathcal{N}_i$.

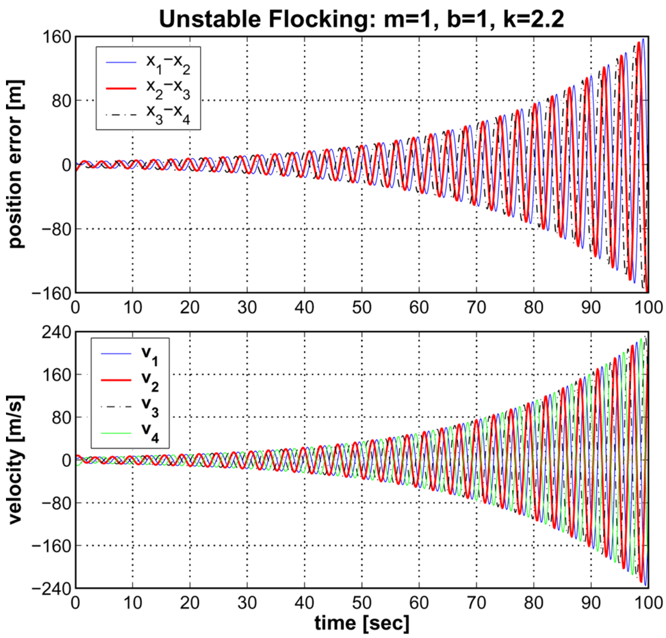

Fig. 2. Unstable flocking on the cyclic graph.

if G is strongly connected, this zero eigenvalue is simple. Because all the eigenvalues of the Laplacian matrix L have nonnegative real parts, the closed-loop group kinematics (3) is stable $\forall k \geq 0$.

In many practical applications (e.g., robots and spacecraft), it is often only possible to directly control the acceleration, not the velocity. Thus, we incorporate the agents' inertias into the flocking model. Similar to (1), let us consider the following closed-loop dynamics of the agent i with its inertia $m_i > 0$

$$m_i \ddot{x}_i = \sum_{j \in \mathcal{N}_i} -b w_{ij}(\dot{x}_i - \dot{x}_j) - k w_{ij}(x_i - x_j) \tag{5}$$

where $b, k > 0$ are the damping and stiffness gains. Then, similar to (3), the closed-loop group dynamics is given by

$$M\ddot{x} + bL\dot{x} + kLx = 0 \tag{6}$$

where $M := \mathrm{diag}[m_1, m_2, \ldots, m_n] \in \Re^{n \times n}$.

If we apply this dynamics-based flocking model (6) for the four agents on the cyclic graph (second graph of Fig. 1) with $(m_i, w_{ij}, b, k) = (1, 1, 1, 2.2)$, we see that the group behavior is unstable as shown in Fig. 2. Note that, if G is undirected as in [12], the dynamics (6) becomes a usual linear time-invariant mass–spring–damper system with a symmetric and positive–semidefinite L, thus, stable for all $b, k \geq 0$. This example clearly shows the importance of the interaction between the agents' inertias and information topology, and the necessity for a framework geared toward agents with nonnegligible inertias and evolving on general directed information graphs.

III. Decomposition

In this section, using the passive decomposition [15], [17], [18], we decompose the closed-loop group dynamics (6) into two systems: a shape system describing the internal group shape and a locked system describing the motion of the center-of-mass.

Following [17] and [18], we define the coordinate transformation

$$z = Sx \tag{7}$$

where $S \in \Re^{n \times n}$ is the (full-rank) transformation matrix defined by

$$S := \begin{bmatrix} \frac{m_1}{\sum_{i=1}^n m_i} & \frac{m_2}{\sum_{i=1}^n m_i} & \frac{m_3}{\sum_{i=1}^n m_i} & \cdots & \frac{m_n}{\sum_{i=1}^n m_i} \\ 1 & -1 & 0 & \cdots & 0 \\ 0 & 1 & -1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 1 & -1 \end{bmatrix}$$

and $z := [z_1, z_2, \ldots, z_n]^T \in \Re^n$ is the transformed coordinate. Let us define $z_e := [z_2, z_3, \ldots, z_n]^T \in \Re^{(n-1)}$ so that $z = [z_1, z_e^T]^T$. Then, from (7), we can show that z_e describes the internal group shape as it is given by

$$z_e = [x_1 - x_2, x_2 - x_3, \ldots, x_{n-1} - x_n]^T \tag{8}$$

and z_1 abstracts the overall group maneuver, as

$$z_1 = \frac{1}{\sum_{i=1}^n m_i}(m_1 x_1 + m_2 x_2 + \ldots + m_n x_n) \tag{9}$$

i.e., the position of the *centroid*. We designate z_1 and z_e as the configurations of the locked and shape systems, respectively, whose dynamics will be defined in the following.

Using (7), we can rewrite the closed-loop group dynamics (6) with respect to z such that

$$S^{-T} M S^{-1} \ddot{z} + bS^{-T} L S^{-1} \dot{z} + kS^{-T} L S^{-1} z = 0 \tag{10}$$

where the inverse of S in (7) is given by [15], [17], [18]

$$S^{-1} = \begin{bmatrix} 1 & \phi_2 & \phi_3 & \cdots & \phi_n \\ 1 & \phi_2 - 1 & \phi_3 & \cdots & \phi_n \\ 1 & \phi_2 - 1 & \phi_3 - 1 & \cdots & \phi_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \phi_2 - 1 & \phi_3 - 1 & \cdots & \phi_n - 1 \end{bmatrix} \tag{11}$$

with $\phi_i := \sum_{k=i}^n m_k / \sum_{j=1}^n m_j$.

Then, using (11), we can show that the inertia matrix M in (10) is block-diagonalized s.t.

$$S^{-T} M S^{-1} =: \mathrm{diag}[m_L, \bar{M}] \tag{12}$$

where $m_L := \sum_{i=1}^n m_i > 0$ and $\bar{M} \in \Re^{(n-1)\times(n-1)}$ is a symmetric and positive–definite matrix. Furthermore, from the property of L that $\sum_{j=1}^n L_{ij} = 0 \, \forall i \in \{1, \ldots, n\}$, the transformed information graph in (10) has the following structure:

$$S^{-T} L S^{-1} = \begin{bmatrix} 0 & \bar{D}^T \\ 0_{(n-1)\times 1} & \bar{L} \end{bmatrix} \tag{13}$$

where the jth component of $\bar{D} \in \Re^{(n-1)}$ is given by

$$\bar{D}_j := -\sum_{k=j+1}^{n} (L_{1k} + L_{2k} + \ldots + L_{nk}) \tag{14}$$

$j \in \{1, 2, \ldots, n-1\}$, and the ijth component of the matrix $\bar{L} \in \Re^{(n-1)\times(n-1)}$ is given by

$$\bar{L}_{ij} = \phi_{i+1}\bar{D}_j + \sum_{p=i+1}^{n}\sum_{q=j+1}^{n} L_{pq} \tag{15}$$

$i, j \in \{1, 2, \ldots, n-1\}$. Proposition 1 is a direct consequence of (7)–(15).

Proposition 1: The closed-loop group dynamics (6) can be decomposed as

$$m_L \ddot{z}_1 + b\bar{D}^T\dot{z}_e + k\bar{D}^T z_e = 0 \tag{16}$$

$$\bar{M}\ddot{z}_e + b\bar{L}\dot{z}_e + k\bar{L}z_e = 0 \tag{17}$$

where $m_L = \sum_{i=1}^{n} m_i > 0$ and $\bar{D}$ and $\bar{L}$ are defined in (14) and (15).

We call the dynamics of z_e (17) *shape system*, which describes the internal group shape as given by (8). We also call the dynamics of z_1 in (16) *locked system*, which represents the centroid motion. Notice from (16) and (17) that the shape system is completely decoupled from the locked system, while the locked system is coupled with the shape system via the coupling term $\bar{D}$.

IV. Flocking on Balanced Graphs

In this section, we consider the flocking problem of multiple inertial agents on strongly connected balanced graphs [8], i.e., any two nodes are connected (i.e., path exists between them) and each node has the same in- and out-degree. See Fig. 1 for examples. We will utilize following properties of balanced graphs.

Proposition 2:

1) The followings are equivalent:
   a) the information graph G is balanced;
   b) all the column sums of L are zeros, i.e., $\sum_{j=1}^{n} L_{ji} = 0$, $\forall i \in \{1, \ldots, n\}$;
   c) the coupling term $\bar{D}$ in (14) vanishes;
   d) the matrix $\bar{L}$ in (15) becomes inertia-independent s.t.

$$\bar{L}_{ij} = \sum_{p=i+1}^{n}\sum_{q=j+1}^{n} L_{pq}. \tag{18}$$

2) If the graph G is balanced and strongly connected, $\bar{L}_{\text{sym}} := (1/2)(\bar{L} + \bar{L}^T)$ is positive definite.

Proof:

1-b) The graph G is balanced, if and only if $\text{in}_i(G) = L_{ii} = \text{out}_i(G) = -\sum_{j=1, j\neq i}^{n} L_{ji}$, $\forall i \in \{1, \ldots, n\}$, where we use the expressions for $\text{in}_i(G)$ and $\text{out}_i(G)$ in Section II. This condition is equivalent to $\sum_{j=1}^{n} L_{ji} = \text{in}_i(G) - \text{out}_i(G) = 0$, $\forall i \in \{1, \ldots, n\}$.

1-c) Following the item 1-b), if the graph G is balanced, all the column sums are zeros. Thus, from (14), $\bar{D}_j = 0$, thus, $\bar{D} = 0$. For the necessity, suppose that $\bar{D} = 0$. Then, $\bar{D}_{n-1} = -\sum_{j=1}^{n} L_{jn} = 0$, i.e., the nth column sum of L should be zero. However, since $\bar{D}_{n-2} = \bar{D}_{n-1} - \sum_{j=1}^{n} L_{j(n-1)}$, if $\bar{D} = 0$, the $(n-1)$th column sum of L should also be zero. By continuing this process up to $\bar{D}_1$, we have $\sum_{j=1}^{n} L_{ji} = 0$ for $i = 2, \ldots, n$ (i.e., all the column sums of L except the first column are zeros). However, since all the row sums of L are zeros (see Section II), $\sum_{i=1}^{n}\sum_{j=1}^{n} L_{ij} = 0$ (i.e., sum of all elements of L is zero). Therefore, the first column sum of L should also be zero. Then, following the item 1-b), G is necessarily balanced.

1-d) This is a direct consequence of the expression (15), the item 1-c), and the fact that $\phi_i > 0$, $\forall i \in \{2, 3, \ldots, n\}$.

2) From (13) with $\bar{D} = 0$ [from item 1-c) of Proposition 2], we can show that $S^{-T} L_{\text{sym}} S^{-1} = \text{diag}[0, \bar{L}_{\text{sym}}]$, where $L_{\text{sym}} := (1/2)(L + L^T)$. Since S^{-1} is full-rank, this is a congruence transform, thus, it preserves the signs of eigenvalues [21, Th. 4.5.8]. Also, if G is balanced and strongly connected, L_{sym} has only one eigenvalue at zero with all the others being strictly positive, since, in this case, L_{sym} is the Laplacian matrix of the (strongly connected and undirected) mirror graph of G [8, Sec. VIII]. Therefore, $\bar{L}_{\text{sym}}$ is positive definite. ■

Therefore, if the graph G is balanced, the centroid dynamics [i.e., locked system in (16)] will be decoupled from the internal group formation [i.e., shape system (17)], and both the locked and shape systems can be analyzed separately. Note that if G is not balanced, such a complete decoupling cannot be ensured.

The difficulty in analyzing the shape system (17) is due to the fact that $\bar{L}$ is generally asymmetric. This difficulty, however, can be overcome for the balanced graphs, by using the fact that $\bar{L}_{\text{sym}}$ is positive definite (item 2 of Proposition 2). We now present the main result.

Theorem 1: Consider the closed-loop group dynamics (6) and their locked and shape dynamics (16) and (17).

1) For arbitrary initial conditions and gains (b, k), the centroid velocity $\dot{z}_1(t)$ is invariant, i.e.,

$$\dot{z}_1(t) = \frac{\sum_{i=1}^{n} m_i \dot{x}_i(t)}{\sum_{i=1}^{n} m_i} = \dot{z}_1(0) \qquad \forall t \geq 0 \tag{19}$$

if and only if the information graph G is balanced.

2) Suppose that the graph G is balanced and strongly connected. Suppose further that we set $b, k > 0$ s.t.

$$b^2\bar{L}_{\text{sym}} - k\bar{M} \succ 0 \tag{20}$$

where $A \succ 0$ implies that $A \in \Re^{n\times n}$ is positive definite. Then, $(\dot{z}_e(t), z_e(t)) \to 0$ exponentially, and

$$\dot{x}_i(t) \to \dot{z}_1(t) = \dot{z}_1(0) = \frac{\sum_{i=1}^{n} m_i \dot{x}_i(0)}{\sum_{i=1}^{n} m_i} \tag{21}$$

$\forall i = 1, \ldots, n$, i.e., the internal group shape converges to the desired one, while all the agents' velocities converge to the invariant centroid velocity.

Proof:

1) Suppose that the graph G is balanced. Then, from Proposition 2, $\bar{D} = 0$ and the locked system dynamics (16) becomes

$$m_L \ddot{z}_1 = 0 \tag{22}$$

where z_1 is defined in (9) and $m_L = \sum_{i=1}^{n} m_i$. Since $\ddot{z}_1(t) = 0\ \forall t \geq 0$ from (22), we have $\dot{z}_1(t) = \dot{z}_1(0), \forall t \geq 0$. For the necessity, suppose that $\dot{z}_1(t) = \dot{z}_1(0), \forall t \geq 0$. Then, from (16), we have $\int_0^t \bar{D}(b\dot{z}_e(\theta) + kz_e(\theta))d\theta = 0\ \forall t \geq 0$, for arbitrary (b, k) and $(z_e(0), \dot{z}_e(0))$. This requires that $\bar{D} = 0$. Following item 1) of Proposition 2, this implies that the graph G is necessarily balanced.

2) Let us consider the shape system dynamics (17), and decompose $\bar{L}$ s.t. $\bar{L} = \bar{L}_{\text{sym}} + \bar{L}_{\text{skew}}$, where $\bar{L}_{\text{skew}} = (1/2)(\bar{L} - \bar{L}^T)$. From item 2) of Proposition 2, $\bar{L}_{\text{sym}}$ is positive definite, since the graph G is strongly connected and balanced.

Let us define a Lyapunov function candidate

$$V := \frac{1}{2}\begin{pmatrix}\dot{z}_e \\ z_e\end{pmatrix}^T P \begin{pmatrix}\dot{z}_e \\ z_e\end{pmatrix} \tag{23}$$

with

$$P := \begin{bmatrix}\bar{M} & \epsilon\bar{M} \\ \epsilon\bar{M} & (k+\epsilon b)\bar{L}_{\text{sym}}\end{bmatrix} \in \Re^{2(n-1)\times 2(n-1)} \tag{24}$$

where $\epsilon > 0$ is to be designed below s.t. P is positive definite. Then, using the shape dynamics (17), and the fact that $v^T \bar{L} w = w^T \bar{L}^T v = (1/2) v^T \bar{L} w + (1/2) w^T \bar{L}^T v$ for all $v, w \in \Re^n$, we can show that

$$\frac{dV}{dt} = -\begin{pmatrix} \dot{z}_e \\ z_e \end{pmatrix}^T Q \begin{pmatrix} \dot{z}_e \\ z_e \end{pmatrix} \tag{25}$$

where $Q \in \Re^{2(n-1)\times 2(n-1)}$ is given by

$$Q := \begin{bmatrix} b\bar{L}_{\mathrm{sym}} - \epsilon \bar{M} & \frac{1}{2}(k-\epsilon b)\bar{L}_{\mathrm{skew}} \\ -\frac{1}{2}(k-\epsilon b)\bar{L}_{\mathrm{skew}} & \epsilon k \bar{L}_{\mathrm{sym}} \end{bmatrix}. \tag{26}$$

Let us choose $\epsilon = k/b$ to cancel out the off-diagonal terms of Q in (26). Then, Q will be positive definite if and only if $b^2 \bar{L}_{\mathrm{sym}} - k\bar{M} \succ 0$. Also, following [21, p. 472], P in (24) with $\epsilon = k/b$ will be positive definite if and only if $2b^2 \bar{L}_{\mathrm{sym}} \succ k\bar{M}$. Thus, if $b^2 \bar{L}_{\mathrm{sym}} - k\bar{M}$ is set to be positive definite as stated in this theorem, both P and Q will be positive definite, and $(\dot{z}_e(t), z_e(t)) \to 0$ exponentially. From (8), it is implied that $(\dot{x}_i(t) - \dot{x}_j(t), x_i(t) - x_j(t)) \to 0$ exponentially $\forall i, j \in \{1, 2, \ldots, n\}$. With $\dot{x}_i(t) \to \dot{x}_j(t)$, from (19), we also have $\dot{z}_1(t) \to \dot{x}_i(t), \forall i = 1, \ldots, n$. Moreover, since the graph G is balanced, following item 1) of this theorem, $\dot{z}_1(t) = \dot{z}_1(0), \forall t \geq 0$. Therefore, we have $\dot{x}_i(t) \to \dot{z}_1(t) = \dot{z}_1(0)$. ■

The condition (20) says that the internal group formation will be stabilized if the damping b and the "good" part of the information topology (i.e., $\bar{L}_{\mathrm{sym}}$) are strong enough, compared to the stiffness gain k and the shape inertia $\bar{M}$. Note from item 1-d) of Proposition 2 that, if the information graph G is balanced, $\bar{L}_{\mathrm{sym}}$ can be directly computed from the graph structure G without knowing agents' inertias, since the expression (18) is inertia-independent. In contrast, the computation of $\bar{M}$ requires only the inertia structure, as its expression in (12) is independent on the information topology.

The condition (20) is simple to use but possibly conservative because of the following: 1) it assumes a specific structure of the Lyapunov function (23) and (24), and 2) it does not utilize any structural information of $\bar{L}_{\mathrm{skew}}$, as its design aims to get rid of the effects of $\bar{L}_{\mathrm{skew}}$ by cancelling out the off-diagonal terms of Q in (26). This conservatism can be partially reduced by directly enforcing $P \succ 0$ and $Q \succ 0$. Note that these two conditions are linear matrix inequalities (LMIs) with respect to (w.r.t.) ϵ. Thus, by solving these LMIs w.r.t. ϵ using available commercial software, we can easily check if a pair of b, k will guarantee stable flocking or can possibly induce unstable behaviors.

Item 1) of Theorem 1 still holds for any (even destabilizing negative) gains b, k and/or arbitrarily fast information topology switching, as long as all the switching graphs are balanced, since, in that case, the dynamics (22) is ensured regardless of the gain-values and/or switchings. On the other hand, since the shape dynamics (17) under the condition (20) is linear and exponentially stable, using the dwell-time argument [19], we can also show that item 2) of Theorem 1 still holds for switching information graphs, if the following are true: 1) all the switching graphs are balanced and strongly connected; 2) the gains (b, k) satisfy the condition (20) uniformly for all the switching graphs; and 3) the switching is slow enough in the sense that the interval between any consecutive switchings is no smaller than a dwell-time $\tau_o > 0$. Following [22], an estimate of this dwell-time can be computed by $\tau_o = (\log \mu)/\lambda_o$ with $\mu := \sup_{p,q\in\mathcal{P}} (\overline{\sigma}[P_p]/\underline{\sigma}[P_q])$ and $\lambda_o := \inf_{p\in\mathcal{P}} (\underline{\sigma}[Q_p]/\overline{\sigma}[P_p])$, where $\mathcal{P}$ is the (finite) switching index set, P_p and Q_p are the P and Q matrices in (23) and (25) associated to the switching index $p \in \mathcal{P}$, and $\overline{\sigma}[P_p]$ and $\underline{\sigma}[P_p]$ are the largest and smallest singular values of P_p, respectively.

For instance, consider the agents under the well-known nearest neighbor rule [9]: Agents i and j can communicate with each other when they are within a sensing radius $r_o > 0$. Then, the following are true: 1) the information graph G is undirected, thus, balanced; 2) we can find a gain pair (b, k) satisfying the condition (20) uniformly for all possible (but only finitely many) switching graphs; and 3) if all agents share a common clock, we can schedule the switching as slow as specified by the aforementioned dwell-time τ_o. Therefore, if we can also enforce connectivity of all the switching graphs (i.e., each is strongly connected), desired flocking behavior can be achieved. This guaranteed connectivity (and slightly less strict joint connectivity) is a standard assumption used in many works (e.g., [8], [9], [11], and [12]). How to enforce such a connectivity for the second-order continuous agents on directed graphs with limited communication range is, in fact, still an open problem and beyond the scope of this note. See [23] and [24] for some preliminary results in this direction.

Remark 1: In Theorem 1, all the agents will reach the position and velocity agreement: $x_i(t) \to x_j(t)$ and $\dot{x}_i(t) \to \dot{x}_j(t)$. Moreover, if all the agents' inertias are the same (i.e., $m_i = m_j \; \forall i, j$), we will achieve velocity average-consensus [8], i.e., $\dot{x}_i(t) \to (1/n)\sum_{i=1}^n \dot{x}_i(0)$.

V. Multidimensional Flocking

For simplicity, we have confined ourselves so far to the scalar flocking problem. The obtained results, however, can be easily extended to a higher dimensional problem as follows. Consider the following multidimensional version of (5):

$$\mathbf{M}_i \ddot{\mathbf{x}}_i = \sum_{j\in\mathcal{N}_i} -b w_{ij} \Lambda(\dot{\mathbf{x}}_i - \dot{\mathbf{x}}_j) - k w_{ij} \Lambda(\mathbf{x}_i - \mathbf{x}_j)$$

where $\mathbf{x}_i \in \Re^m$ is the position, $\mathbf{M}_i, \Lambda \in \Re^{m\times m}$ are the symmetric and positive–definite inertia and gain matrices, respectively, and the other terms are defined the same as in (5). Then, similar to (6), we can stack them up to achieve the following group dynamics:

$$\mathbf{M}\ddot{\mathbf{x}} + b(L\otimes\Lambda)\dot{\mathbf{x}} + k(L\otimes\Lambda)\mathbf{x} = 0$$

where $\mathbf{x} := [\mathbf{x}_1^T, \mathbf{x}_2^T, \ldots, \mathbf{x}_n^T] \in \Re^{nm}$, $\mathbf{M} := \mathrm{diag}[\mathbf{M}_1, \mathbf{M}_2, \ldots \mathbf{M}_n] \in \Re^{nm\times nm}$, $L \in \Re^{n\times n}$ is the Laplacian matrix of G, and $\otimes$ is the Kronecker product [25].

Following [17] and [18], the multidimensional version of the passive decomposition matrices $\mathbf{S}, \mathbf{S}^{-1} \in \Re^{nm\times nm}$ are given by replacing the scalars m_i/m_L, 1, and $\phi_i = \sum_{k=i}^n m_k/m_L$ of (7) and (11) with the $m\times m$ matrices $\mathbf{M}_L^{-1}\mathbf{M}_i, \mathbf{I}$, and $\boldsymbol{\phi}_i := \mathbf{M}_L^{-1}\sum_{k=i}^n \mathbf{M}_k$, respectively, where $\mathbf{M}_L = \sum_{i=1}^n \mathbf{M}_i \in \Re^{m\times m}$. These $\mathbf{S}$ and $\mathbf{S}^{-1}$ give us similar expressions as (12) and (13), and, eventually, we can achieve the locked and shape dynamics similar to (16) and (17) s.t.

$$\mathbf{M}_L \ddot{\mathbf{z}}_1 + b\bar{\mathbf{D}}^T \dot{\mathbf{z}}_e + k\bar{\mathbf{D}}^T \mathbf{z}_e = 0 \tag{27}$$

$$\bar{\mathbf{M}}\ddot{\mathbf{z}}_e + b\bar{\mathbf{L}}\dot{\mathbf{z}}_e + k\bar{\mathbf{L}}\mathbf{z}_e = 0 \tag{28}$$

where $\mathbf{z}_1 = \mathbf{M}_L^{-1}\sum_{i=1}^n \mathbf{M}_i \mathbf{x}_i \in \Re^m$, $\mathbf{z}_e = [\mathbf{x}_1^T - \mathbf{x}_2^T, \ldots, \mathbf{x}_{n-1}^T - \mathbf{x}_n^T] \in \Re^{(n-1)m}$, $\bar{\mathbf{D}} := \bar{D}\otimes\Lambda \in \Re^{(n-1)m}$ with its jth block-component given by

$$\bar{\mathbf{D}}_j = \bar{D}_j \otimes \Lambda = \bar{D}_j \Lambda \in \Re^{m\times m} \tag{29}$$

[see (14) for definition of $\bar{D}$], $\bar{\mathbf{L}} \in \Re^{(n-1)m\times(n-1)m}$ has its the ijth block-component given by

$$\bar{\mathbf{L}}_{ij} = \boldsymbol{\phi}_{i+1}^T \bar{\mathbf{D}}_j + \sum_{p=i+1}^n \sum_{q=j+1}^n L_{pq}\Lambda \in \Re^{m\times m} \tag{30}$$

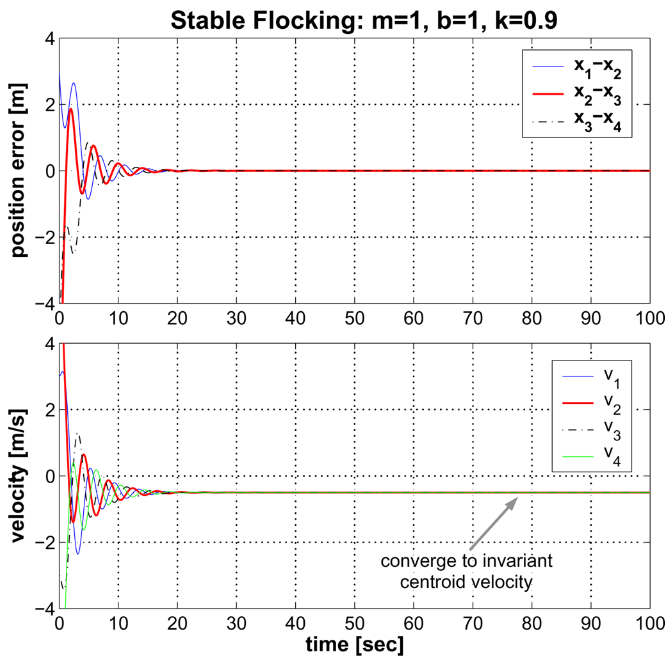

Fig. 3. Stable flocking on the cyclic graph with the condition (20).

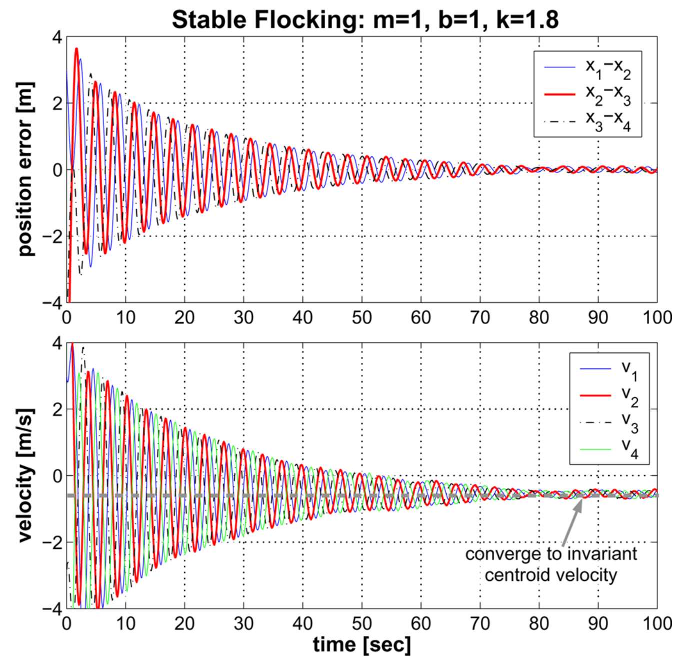

Fig. 4. Stable flocking on the cyclic graph with LMI conditions $P \succ 0, Q \succ 0$.

and $\bar{\mathbf{M}} \in \Re^{(n-1)m\times(n-1)m}$ is the symmetric and positive–definite shape inertia matrix.

Here, using (29) and (30), we can easily show that Proposition 2 still hold for this multidimensional case as well. For instance, from (29), $\bar{\mathbf{D}}$ will vanish if and only if the graph G is balanced, since the vanishing of $\bar{\mathbf{D}} \in \Re^{(n-1)m}$ is equivalent to that of $\bar{D} \in \Re^{n-1}$. Also, if the graph G is balanced, $\bar{\mathbf{D}}$ vanishes, thus, from (30) and (15), we have $\bar{\mathbf{L}} = \bar{L} \otimes \Lambda$. Then, from the property of the Kronecker product [25], we can show that $\bar{\mathbf{L}}_{\rm sym} = \bar{L}_{\rm sym} \otimes \Lambda$ and its eigenvalues are given by $\alpha_i\beta_j$, where α_i and β_j are, respectively, the ith and jth eigenvalues of $\bar{L}_{\rm sym}$ and Λ. Here, $\beta_j > 0\ \forall j = 1,\ldots,m$, since Λ is symmetric and positive definite. Also, if the graph G is strongly connected, $\alpha_i > 0\ \forall i = 1,\ldots,n-1$ (from Proposition 2). Therefore, if G is balanced and strongly connected, $\bar{\mathbf{L}}_{\rm sym}$ is positive definite.

With this multidimensional version of Proposition 2, we can also easily extend the main Theorem 1 to the higher dimension problem. First, for the item 1) of Theorem 1, if the graph G is balanced, from (27) with $\bar{\mathbf{D}} = 0$, we have $\mathbf{M}_L\ddot{\mathbf{z}}_1 = 0$. Thus, the centroid velocity $\dot{\mathbf{z}}_1(t) = \mathbf{M}_L^{-1}\sum_{i=1}^{n}\mathbf{M}_i\dot{\mathbf{x}}_i(t) \in \Re^m$ is invariant. On the other hand, if G is not balanced, $\bar{\mathbf{D}} \neq 0$. Thus, for some initial conditions $(\dot{\mathbf{z}}_e(0), \mathbf{z}_e(0))$ and gain pair (b,k), $\mathbf{M}_L\ddot{\mathbf{z}}_1(t) \neq 0$ during some time, therefore, $\dot{\mathbf{z}}_1(t)$ is no longer invariant. For the extension of item 2) of Theorem 1, we can use the same Lyapunov argument: With the same $\epsilon = k/b$, and by replacing $\bar{M}$ and $\bar{L}_{\rm sym}$ in (24) and (26) with their multidimensional counterparts $\bar{\mathbf{M}}$ and $\bar{\mathbf{L}}_{\rm sym}$, we can show that $(\dot{\mathbf{z}}_e(t), \mathbf{z}_e(t)) \rightarrow 0$ exponentially and $\dot{\mathbf{x}}_i(t) \rightarrow \dot{\mathbf{z}}_1(t) = \dot{\mathbf{z}}_1(0)$, if the graph G is balanced and strongly connected and the gains (b,k) are set to be $b^2\bar{\mathbf{L}}_{\rm sym} - k\bar{\mathbf{M}} \succ 0$. Similarly, the slow-switching argument and Remark 1 in Section IV are also valid for the multidimensional case.

VI. Simulation

We apply the results of Theorem 1 to the example of Section II (i.e., four agents on the cyclic graph). As shown in Fig. 3, with the condition (20), the group behavior now becomes stable. Also, the desired internal group shape is achieved (top of Fig. 3), and individual agents' velocities converge to the invariant centroid velocity (bottom of Fig. 3). Since we set all the agents' inertias to be the same, as stated in Remark 1, the velocity average-consensus is also achieved.

In Fig. 4, we use the less conservative LMI conditions stated in the second paragraph after Theorem 1. The desired internal group shape and the velocity average-consensus are still achieved. However, as the design becomes less conservative, the group behavior becomes more oscillatory than that in Fig. 3. This LMI-based condition becomes infeasible when $k \geq 2$ with $m_i = w_{ij} = b = 1$. This well matches with the fact that the group behavior becomes unstable for k larger than 2 with $(m_i, w_{ij}, b) = (1,1,1)$ (e.g., Fig. 2).

VII. Conclusion

In this note, we consider the multiagent flocking problem. We first show that, for a certain directed information graph, the group behavior can go unstable, if the agents' inertial effect is neglected. Then, we propose a novel provably stable flocking framework for multiple agents which have significant inertial effects and evolve on balanced information graphs. By relying on passive decomposition and graph theory, the proposed framework ensures that the internal group shape is exponentially stabilized to a desired one, while the velocities of all the agents converge to the (time-invariant) centroid velocity.

There are several research directions we will pursue in future. The gain-setting condition (20) (or the LMI condition) is only sufficient for stability. Since the matrices in the decomposed dynamics possess specific structures and properties [e.g., (18)], using matrix theory, we may be able to exploit such properties/structure to reduce the conservatism. Relying on the dwell-time concept [19], we show that the proposed framework is also applicable for slow-switching balanced graphs. However, it is not yet clear how each system parameter affects the dwell-time, as they are all mingled together in the system dynamics (17). Of particular interest is to quantify the effect of the information topology on the dwell time. We would also like to extend this framework to the case where information topology and agents' motion are coupled with each other and/or their motions are required not to collide with each other.

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Choosing Between Open- and Closed-Loop Experiments in Linear System Identification

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***Abstract*—This correspondence shows that open-loop experiments are optimal for a broad class of systems when the system input is constrained. In addition, we show that, for a general class of systems, when the output power is constrained, closed-loop experiments are optimal. Both results use a strong notion of optimality and use expressions for estimation accuracy which are nonasymptotic in model order but asymptotic in data length.**

***Index Terms*—Closed-loop systems, identification, input design.**

I. Introduction

Experiment design has received substantial attention in both statistics (see [1]–[4] and the references therein) and engineering (see, for example, [5] and [6]). Choosing between open- and closed-loop experiments has been a recurring question in this area [5]–[11].

Previous relevant results include the following: One of the earliest references to the benefits of closed-loop experiments appears in [5] and [7] where it was shown that, when the output power is constrained, then a closed-loop experiment is optimal for a special $\{\mathrm{ARX}\}$ model structure. In [8], an extension of this result was presented for indirect identification. Later, in [9] and [10] it was shown that closed-loop experiments were optimal (when the output power is constrained) for a particular cost function and using an asymptotic (in model order) expression for estimation accuracy. Also in [11], it was shown that closed-loop experiments were optimal when one constrains the part of the input arising from the reference signal. (Also see embellishments in [12, Lemma 2].)

In this correspondence, we present new insights into this problem which extend results recently presented in [12]. Core departures from earlier work include: i) we work with general Box-Jenkins (BJ) models; ii) we compare experiment designs by using a strong notion of optimality; and iii) we use expressions for model accuracy that are asymptotic in data length but non-asymptotic in model order.

Our key results are that (under quite general conditions): i) open-loop experiments are strongly optimal (for a broad class of systems) when the system input is constrained and, ii) closed-loop experiments are strongly optimal (for a general class of systems) when the output power is constrained.

II. System Description and Background

Consider a single-input–single-output (SISO) linear system of the form

$$\mathcal{S} = \left\{(G_o, H_o) \in \mathcal{C} : y(t) = G_o(q^{-1})u(t) + H_o(q^{-1})w(t)\right\} \quad (1)$$

where $\mathcal{C}$ is the set of causal linear systems, q^{-1} is the unit delay operator, and[1] $G_o(q^{-1}) = q^{-d}\bar{G}_o(q^{-1})$ $(\bar{G}(0) = b_0 \neq 0, d \in \mathbb{N})$ and

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[1]We take $\bar{G}_o(q^{-1}) = (\bar{B}_o(q^{-1}))/(A_o(q^{-1}))$.