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Passive Configuration Decomposition and Practical Stabilization of Nonholonomic Mechanical Systems with Symmetry

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***Abstract*— We consider a certain class of nonholonomic mechanical systems with symmetry. First, we introduce the notion of passive configuration decomposition, which enables us to decompose the system's Lagrange-D'Alembert dynamics into two decoupled systems, evolving on their respective configuration spaces according to the symmetry and individually possessing Lagrangian-like structure and passivity. We then propose passivity-based time-varying and switching control laws, which achieve practical stabilization by controlling the two decoupled systems on their own configuration spaces individually. Multiple wheeled mobile robots formation stabilization simulation is also performed to support the theory.**

I. INTRODUCTION

In this paper, we consider the practical stabilization problem [1] of a certain class of nonholonomic mechanical systems with symmetry. More specifically, inspired by [2], [3], [4], we consider the nonholonomic mechanical systems with the following properties: 1) its configuration space $\mathcal{M}$ has the natural product structure $\mathcal{M} = \mathcal{S} \times \mathcal{R}$ with its configuration given by $q = (s, r) \in \mathcal{M}$, $s \in \mathcal{S}, r \in \mathcal{R}$; 2) its inertia metric $M(r)$ is a function of only $r \in \mathcal{R}$, not $s \in \mathcal{S}$; 3) its nonholonomic Pfaffian constraint is a function of only $r \in \mathcal{R}$, yet, acts only on $\mathcal{S}$; and 4) it has no geometric phase. We also assume that the system's Lagrangian is solely given by its kinetic energy.

Then, using our recently proposed nonholonomic passive decomposition [5], [6], [7], we show that, for the nonholonomic mechanical systems as described above, its Lagrange-D'Alembert dynamics can be decomposed into the two decoupled systems: 1) ν_s-dynamics on $\mathcal{S}$, where ν_s is the portion of $\dot{s}$ respecting the nonholonomic constraint on $\mathcal{S}$; and 2) $\dot{r}$-dynamics on $\mathcal{R}$. These two decoupled dynamics individually possess Lagrangian-like structure and (open-loop) passivity. Their energetics are also decoupled from each other. Therefore, we can essentially think of the system's total motion as composed of the ν_s-dynamics on $\mathcal{S}$ and the $\dot{r}$-dynamics on $\mathcal{R}$, with their kinetic energies also decomposed into $\mathcal{S}$ and $\mathcal{R}$. Due to this reason, we name our decomposition **passive configuration decomposition**.

These Lagrangian-like structure and passivity of the ν_s-dynamics and the $\dot{r}$-dynamics, inherited from the original Lagrange-D'Alembert dynamics, greatly facilitate their control design on $\mathcal{S}$ and $\mathcal{R}$ (e.g. many standard passivity-based control techniques are directly applicable for them). Utilizing this property, we then provide two passivity-based control laws to achieve the practical stabilization in the sense of [1] (i.e. q stabilizes to a small neighborhood of q_d): 1) *passivity-based time-varying control*, for which we modify the formation control law of [6] to opportunistically utilize the system's symmetry; and 2) *passivity-based switching control*, for which we extend the results of [8] (only for unicycles in SE(2)) to the (more general) nonholonomic systems as stated above while providing complete analysis, that was only briefly alluded in [8].

There are many powerful control techniques available for kinematic nonholonomic systems [9]. Yet, feedback control techniques for (dynamic) nonholonomic mechanical systems are rare, although there are many strong results on their (open-loop) optimal control (e.g. [3]) and controllability analysis (e.g. [10], [11]). Even rarer are the feedback control techniques, that exploit the open-loop dynamics' Lagrangian structure and passivity rather than cancelling them out to achieve kinematic equations (e.g. normal form [12] -see [6], [7], [13], [14] for some notable exceptions). In contrast, our feedback stabilization control frameworks proposed here aim to preserve/utilize these Lagrangian structure and passivity, which have been the key concepts for many control techniques for unconstrained mechanical systems, yet, surprisingly missing for nonholonomic mechanical systems so far.

The rest of this paper is organized as follows. After introducing some preliminary materials in Sec. II, passive configuration decomposition is proposed in Sec. III. Passivity-based time-varying and switching control laws are then designed in Sec. IV along with simulation results on multiple wheeled mobile robot (WMR) stabilization. Some concluding remarks are then given in Sec. V.

II. PRELIMINARY

Let us start with the dynamics of nonholonomic mechanical systems, consisting of 1) nonholonomic Pfaffian constraint:

$$A(q)\dot{q} = 0 \tag{1}$$

and 2) Lagrange-D'Alembert equation of motion:

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + A^T(q)\lambda = \tau + f \tag{2}$$

where $q, \dot{q}, \tau, f \in \Re^n$ are the configuration, velocity, control, and external force, $M, C \in \Re^{n\times n}$ are the inertia and Coriolis matrices with $\dot{M} - 2C$ being skew-symmetric, $A(q) \in \Re^{p\times n}$ ($p \leq n$) defines the nonholonomic constraint, and $A^T(q)\lambda$ is the constraint force, whose magnitude is specified by the Lagrange multiplier $\lambda \in \Re^p$. Here, we (locally) identify the system's configuration space $\mathcal{M}$ by $\Re^n$ (i.e. $\mathcal{M} \approx \Re^n$). We also assume that the nonholonomic constraint (1) is smooth

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and regular (i.e. $\operatorname{rank} A(q) = p$ for all q). This modeling (1)-(2) is applicable to multiple nonholonomic systems as well, by combining their individual dynamics and constraints into their product configuration space.

We also assume the following symmetry structure: 1) the system's configuration space $\mathcal{M}$ has the product structure s.t.

$$\mathcal{M} = \mathcal{S} \times \mathcal{R}$$

with $q := [s; r]$, $s \in \Re^{n-m}$ and $r \in \Re^m$, and $\tau = [\tau_s; \tau_r]$, where we use the notation $[x; y] := [x^T, y^T]^T$; 2) its inertia metric is a function of only $r \in \mathcal{R}$, i.e. $M(q) = M(r)$ with $C(q, \dot{q}) = C(r, \dot{q})$; and 3) the nonholonomic Pfaffian constraint is also a function of only $r \in \mathcal{R}$ and acts only on $\mathcal{S}$, i.e.

$$A(q)\dot{q} = \begin{bmatrix} A_s(r) & 0_{p\times m} \end{bmatrix} \dot{q} = A_s(r)\dot{s} = 0$$

with $A_s(r) \in \Re^{p\times(n-m)}$ is full row-rank. This class of systems is slightly more restricted than that of [2], [3], [4], since, here, $A(r)$ acts only on $\mathcal{S}$, yet, there, $A(r)$ acts both on $\mathcal{R}$ and $\mathcal{S}$. Even so, this class of systems still encompasses many practically important and interesting systems (e.g. differential WMR [8], mobile manipulator, beanie [4], hopping robot [15], vertical coin [16], etc).

Then, the unconstrained distribution is given by:

$$\mathcal{D} := \operatorname{span} \begin{bmatrix} \mathcal{D}_s(r) & 0 \\ 0 & I_{m\times m} \end{bmatrix} \tag{3}$$

where $\operatorname{span} \star$ is the span of the column vectors of $\star$, $\mathcal{D}_s(r) \in \Re^{(n-m)\times(n-m-p)}$ defines the unconstrained distribution on $\mathcal{S}$ s.t. $A_s(r)\mathcal{D}_s(r) = 0$, and we can write $\dot{q}$ s.t.

$$\dot{q} = \begin{pmatrix} \dot{s} \\ \dot{r} \end{pmatrix} = \underbrace{\begin{bmatrix} \mathcal{D}_s(r) & 0 \\ 0 & I_{m\times m} \end{bmatrix}}_{=:S'(r)\in\Re^{n\times(n-p)}} \begin{pmatrix} \nu_s' \\ \dot{r} \end{pmatrix}$$

where $\nu_s' \in \Re^{n-m-p}$ encodes the (permissible) velocity on $\mathcal{S}$ respecting the nonholonomic constraint. Also, note that, due to the symmetry, $S'(r)$ is only a function of r.

III. Passive Configuration Decomposition

We may rewrite the dynamics (2) w.r.t. ν_s' and $\dot{r}$. This, yet, in general, comes with acceleration coupling between ν_s'-dynamics and $\dot{r}$-dynamics. To avoid this, here, we utilize nonholonomic passive decomposition of [5], [6], [7] to achieve passive configuration decomposition. For this, define $h(q) := r$ (i.e. formation map); and denote the null-space of $\partial h/\partial q$ by $\Delta^\top \approx T_s\mathcal{S}$, and its orthogonal complement w.r.t. M-metric by $\Delta^\perp$. Since the nonholonomic constraint is all contained in $T_s\mathcal{S}$, the motion on $T_s\mathcal{S}$ is permitted only within $\mathcal{D} \cap \Delta^\top$, while that in $\Delta^\perp$ free from the constraint with $\mathcal{D} \cap \Delta^\perp = \Delta^\perp$. See Fig. 1.

We can then write $\dot{q}$ s.t.

$$\dot{q} = \begin{pmatrix} \dot{s} \\ \dot{r} \end{pmatrix} = \underbrace{\begin{bmatrix} \mathcal{D}_s(r) & \beta(r) \\ 0 & I_{m\times m} \end{bmatrix}}_{=:S(r)\in\Re^{n\times(n-p)}} \begin{pmatrix} \nu_s \\ \dot{r} \end{pmatrix} \tag{4}$$

where

$$\beta(r) := -\mathcal{D}_s(r)[\mathcal{D}_s^T(r)M_1(r)\mathcal{D}_s(r)]^{-1}\mathcal{D}_s^T(r)M_2(r) \tag{5}$$

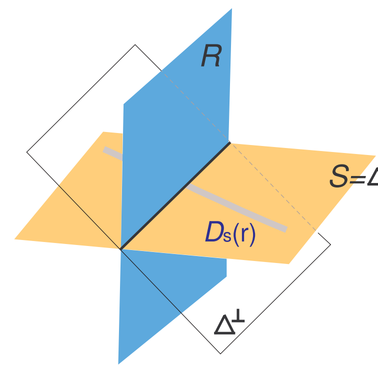

Fig. 1. Illustration of $\mathcal{S} \approx \Delta^\top$, $\mathcal{R}$, $\mathcal{D}_s$ and $\Delta^\perp$.

with the inertia $M(r)$ written as

$$M(r) =: \begin{bmatrix} M_1(r) & M_2(r) \\ M_2^T(r) & M_3(r) \end{bmatrix}$$

Here, $\operatorname{span}[\mathcal{D}_s; 0]$ and $\operatorname{span}[\beta(r); I]$ in $S(r)$ respectively identify $\mathcal{D} \cap \Delta^\top$ and $\Delta^\perp$, while ν_s and $\dot{r}$ representing the velocity components in each of them. We also assume $\mathcal{D}_s^T\mathcal{D}_s = I$, which is always possible since $\mathcal{D}_s$ has full column rank.

Note that, with $\mathcal{D}_s$ in front of (5), we have $\operatorname{span} S(r) = \operatorname{span} S'(r) = \mathcal{D}$. This choice of $\beta(r)$ in (5) also enforces M-orthogonality as shown by that

$$S^T M S = \begin{bmatrix} \mathcal{D}_s^T M_1 \mathcal{D}_s & 0 \\ 0 & \beta^T M_1 \beta + \beta^T M_2 + M_2^T \beta + M_3 \end{bmatrix}$$

where the top and bottom non-zero terms are respectively the inertia $M(r)$ projected on $\mathcal{D} \cap \Delta^\top$ and $\Delta^\perp$. Note also that the inverse in (5) always exists, since $M_1(r)$ is invertible and $\mathcal{D}_s(r)$ has full column-rank. This $\beta(r)$ in fact defines *nonholonomic connection* [2].

Since $\Delta^\perp$ is $M(q)$-orthogonal w.r.t. $\Delta^\top(\approx \mathcal{S})$, $\Delta^\perp$ is in general not parallel w.r.t. $T_r\mathcal{R}$ (i.e. $\beta(r) \neq 0$ in (4)). If parallel, we may then decompose the Lagrange-D'Alembert dynamics into those on $\mathcal{S}$ and on $\mathcal{R}$ - see Th. 1. Here, also, $\Delta^\perp \subset \mathcal{D}$ implies strong decomposability [5], [6], [7], under which the Lagrange-D'Alembert dynamics (2) can be exactly decomposed into the ν_s-dynamics and $\dot{r}$-dynamics without any remaining (quotient) dynamics.

Applying (4), we can then decompose the Lagrange-D'Alembert dynamics (2) s.t.

$$D_s(r)\dot{\nu}_s + Q_s(r,\dot{q})\nu_s + Q_{sr}(r,\dot{q})\dot{r} = u_s + \delta_s \tag{6}$$

$$D_r(r)\ddot{r} + Q_r(r,\dot{q})\dot{r} + Q_{rs}(r,\dot{q})\nu_s = u_r + \delta_r \tag{7}$$

where (6) is the projected dynamics on $\mathcal{D} \cap \Delta^\top$ while (7) that on $\Delta^\perp$, $\operatorname{diag}[D_s, D_r] := S^T M S$,

$$\begin{bmatrix} Q_s & Q_{sr} \\ Q_{rs} & Q_r \end{bmatrix} := S^T \left[M\dot{S} + CS \right] \tag{8}$$

and

$$\begin{pmatrix} u_s \\ u_r \end{pmatrix} := S^T(r) \begin{pmatrix} \tau_s \\ \tau_r \end{pmatrix} = \begin{pmatrix} \mathcal{D}_s^T \tau_s \\ \beta^T(r)\tau_s + \tau_r \end{pmatrix} \tag{9}$$

are the projected controls (similar also holds for $f_\star$ and $\delta_\star$).

Proposition 1 *Consider the nonholonomic mechanical system* (1)-(2) *with the symmetry. Then, we can decompose its Lagrange-D'Alembert dynamics* (2) *into* (6)-(7)*, where*

1) *D_s and D_r are symmetric and positive-definite.*
2) *$\dot{D}_s - 2Q_s$ and $\dot{D}_r - 2Q_r$ are skew-symmetric.*
3) *$Q_{sr} = -Q_{rs}^T$.*
4) *Kinetic energy and power are decomposed s.t.*

$$\kappa(t) = \frac{1}{2}\nu_s^T D_s \nu_s + \frac{1}{2}\dot{r}^T D_r \dot{r}, \quad \tau^T \dot{q} = u_s^T \nu_s + u_r^T \dot{r}$$

(similar also hold for f and δ), where $\kappa := \dot{q}^T M \dot{q}/2$.

Proof: See [6]. ∎

From (4), we can also write $\dot{s}$ by

$$\dot{s} = \mathcal{D}_s(r)\nu_s + \beta(r)\dot{r} \tag{10}$$

where the first and second terms are respectively called *dynamic phase* and *geometric phase* [2]. If the dynamic phase vanishes with $\dot{s} = \beta(r)\dot{r}$, it is called *principal kinematic case* [4]. Here, note that $\beta(r)\dot{r}$ is the velocity component shown in $T_s\mathcal{S}$ as the result of the $\dot{r}$-motion in $\Delta^\perp$, since $\Delta^\perp$ is generally not aligned with $T_r\mathcal{R}$ as shown in Fig. 1.

If $\beta(r) = 0$ in (10) (i.e. geometric phase vanishing), we then have, from (4), $\Delta^\perp \approx T_r\mathcal{R}$, the artifact component $\beta(r)\dot{r}$ in (10) disappear, and the motions on $\mathcal{S}$ and $\mathcal{R}$ can be respectively described by $\dot{s} = \mathcal{D}_s(r)\nu_s$ and $\dot{r}$. Thus, in this case, we can essentially think of the system's total motion on $\mathcal{M}$ as to be composed of the ν_s-dynamics (6) on $\mathcal{S}$; and the $\dot{r}$-dynamics (7) on $\mathcal{R}$. Due to this reason, if $\beta(r) = 0$, we call the decomposition **passive configuration decomposition** into $\mathcal{S}$ and $\mathcal{R}$, as formalized in the below theorem, which shows that the kinetic energy is also decomposed into those in $\mathcal{S}$ and $\mathcal{R}$.

Theorem 1 *If $\beta(r) = 0$, the nonholonomic mechanical system with the symmetry* (1)-(2) *can be decomposed into the ν_s-dynamics* (6) *on $\mathcal{S}$ (with $\dot{s} = \mathcal{D}_s(r)\nu_s$) and the r-dynamics* (7) *on $\mathcal{R}$, with the kinetic energy and power also decomposed into those of $\mathcal{S}$ and $\mathcal{R}$ s.t.*

$$\frac{1}{2}\nu_s^T D_s \nu_s = \frac{1}{2}\dot{s}^T M_1 \dot{s} \quad \textit{and} \quad u_s^T \nu_s = \tau_s^T \dot{s} \quad \textit{on } \mathcal{S} \tag{11}$$

$$\frac{1}{2}\dot{r}^T D_r \dot{r} = \frac{1}{2}\dot{r}^T M_3 \dot{r} \quad \textit{and} \quad u_r^T \dot{r} = \tau_r^T \dot{r} \quad \textit{on } \mathcal{R} \tag{12}$$

Proof: Here, we only prove power parts in (11)-(12), since others are easy to show. From (9) and (10), we have

$$u_s^T \nu_s = \tau_s^T \mathcal{D}_s \nu_s = \tau_s^T [\dot{s} - \beta(r)\dot{r}] \tag{13}$$

$$u_r^T \dot{r} = [\beta^T(r)\tau_s + \tau_r]^T \dot{r} \tag{14}$$

which become (11)-(12) with $\beta(r) = 0$. ∎

The following lemma provides the necessary and sufficient conditions for $\beta(r) = 0$ (i.e. geometric phase vanishing).

Lemma 1 *Geometric phase vanishes with $\beta(r) = 0$, if and only if $M_2(r) = 0$ or $M_2^T(r)\mathcal{D}_s(r) = 0$ for all r.*

Proof: It is easy to see from (5) that $\beta(r) = 0$, if $M_2 = 0$ or $M_2^T \mathcal{D}_s = 0$ (i.e. $\mathcal{D}_s^T M_2 = 0$). Also, suppose $\beta(r) = 0$. Then, from the kinetic energy decomposition in Prop. 1 with (11), (12), and $\beta(r) = 0$,

$$\begin{aligned}\kappa(t) &= \frac{1}{2}\dot{s}^T M_1 \dot{s} + \frac{1}{2}\dot{r}^T M_3 \dot{r} + \dot{r}^T M_2^T \dot{s} \\ &= \frac{1}{2}\nu_s^T \mathcal{D}_s \nu_s + \frac{1}{2}\dot{r}^T \mathcal{D}_r \dot{r} = \frac{1}{2}\dot{s}^T M_1 \dot{s} + \frac{1}{2}\dot{r}^T M_3 \dot{r}\end{aligned}$$

for all permissible $\dot{s}$ and $\dot{r}$. This then implies that $\dot{r}^T M_2^T \dot{s} = \dot{r}^T M_2^T \mathcal{D}_s \nu_s = 0$, for any $\dot{r} \in \Re^m$ and $\nu_s \in \Re^{n-m-p}$, which can only be true when $M_2^T \mathcal{D}_s = 0$ or $M_2 = 0$. ∎

Once we achieve this passive configuration decomposition, we can then drive the system's motion on $\mathcal{M}$ by individually controlling the motion on $\mathcal{S}$ and that on $\mathcal{R}$. This is further facilitated by the Lagrangian-like structure and passivity of the decomposed dynamics (6)-(7). These properties will be utilized in the next section to achieve feedback practical stabilization on $\mathcal{M}$.

IV. Passivity-Based Practical Stabilization

For the control design, we assume that the nonholonomic mechanical system (1)-(2) has full control for the motions in $\mathcal{D}$. This means that u_r, u_s in (9) can be arbitrarily assigned. The issue of under-actuation within $\mathcal{D}$ itself defines a challenging problem and we spare it for future research.

We consider the stabilization problem, that is, $q = (s, r) \to (s_d, r_d) =: q_d$, where s_d and r_d are the desired constant set-points defined respectively on $\mathcal{S}$ and $\mathcal{R}$. To achieve $r \to r_d$ is not difficult since the $\dot{r}$-dynamics (7) on $\mathcal{R}$ has the structure of unconstrained Lagrangian systems. Yet, that is not the case for the s-stabilization, since the motion in $\mathcal{S}$ is restricted by the nonholonomic constraint (1). Even so, its Lagrangian-like structure and passivity still suggests a potential field as one viable option. More precisely, we define a non-negative navigation potential

$$\varphi_s : \mathcal{S} \to \Re$$

s.t. 1) $\varphi_s(s) \geq 0$ with the equality hold only when $s = s_d$; 2) $\partial\varphi_s/\partial s(s) := [\partial\varphi_s/\partial s_1, ..., \partial\varphi_s/\partial s_{n-m}] = 0$ iff $s = s_d$; and 3) for any $l \geq 0$, the level set

$$\mathcal{L}_l := \{s \in \mathcal{S} \mid \varphi_s(s) \leq l\} \tag{15}$$

is a compact set containing $s = s_d$ and $\mathcal{L}_{l_1} \subseteq \mathcal{L}_{l_2}$ if $l_2 \geq l_1 \geq 0$. This φ_s may also incorporate other objectives (e.g. avoidance). Note here that this φ_s is designed on $\mathcal{S}$ as if there is no nonholonomic constraint on $\mathcal{S}$. Thus, we can use many standard potential functions for φ_s.

Incorporating this φ_s, we design the control u_s for (6) s.t.

$$u_s = Q_{sr}(r, \dot{q})\dot{r} - b_s\nu_s - \mathcal{D}_s^T(r)\left[\frac{\partial\varphi_s(s)}{\partial s}\right]^T - \delta_s \tag{16}$$

where $b_s \in \Re^{(n-m-p)\times(n-m-p)}$ is positive-definite symmetric damping gain, $Q_{sr}\dot{r}$ is the decoupling control, and

$\partial\varphi_s/\partial s \in \Re^{1\times(n-m-p)}$ is the one-form of $\varphi_s(s)$. Then, the closed-loop ν_s-dynamics (6) becomes

$$D_s(r)\dot{\nu}_s + Q_s(r,\dot{q})\nu_s + b_s\nu_s + \mathcal{D}_s^T(r)\left[\frac{\partial\varphi_s(s)}{\partial s}\right]^T = 0. \tag{17}$$

Using Prop. 1 with (10), we can also show that

$$\begin{aligned}\frac{d\kappa_s}{dt} &= -||\nu_s||_{b_s}^2 - \frac{\partial\varphi_s}{\partial s}\mathcal{D}_s\nu_s \\ &= -||\nu_s||_{b_s}^2 - \frac{d\varphi_s(s)}{dt} + \frac{\partial\varphi_s(s)}{\partial s}\beta(r)\dot{r}\end{aligned}$$

where $\kappa_s := \nu_s^T\mathcal{D}_s\nu_s/2$, $||\nu_s|_{b_s}^2 := \nu_s^T b_s\nu_s$, and $d\varphi_s/dt = [\partial\varphi_s/\partial s]\dot{s}$. Integrating this with $\beta(r) = 0$, we further have:

$$V_s(T) - V_s(0) = -\int_0^T ||\nu_s||_{b_s}^2 dt, \qquad \forall T \geq 0 \tag{18}$$

where $V_s(t) := \kappa_s(t) + \varphi_s(t)$. Thus, if $\beta(r) = 0$, the closed-loop ν_s-dynamics (17) and its energetics (18) become very similar to those of usual (unconstrained) mechanical systems. The following lemma is a consequence of this observation.

Lemma 2 *Consider the ν_s-dynamics* (6) *with u_s in* (16) *and $\beta(r) = 0$. Suppose further that: 1) partial derivatives of $M(r)$ w.r.t. r of any order are bounded; 2) partial derivatives of $\varphi_s(s)$ w.r.t. s of any order are bounded if $\varphi_s(s)$ is bounded; and 3) $\dot{r}, \ddot{r} \in \mathcal{L}_\infty$. Then, $\nu_s \to 0$ and*

$$\mathcal{D}_s^T(r)\left[\frac{\partial\varphi_s(s)}{\partial s}\right]^T \to 0 \tag{19}$$

Proof: First, with $\beta(r) = 0$, from (18), we have $\nu_s \in \mathcal{L}_\infty \cap \mathcal{L}_2$. This also means $\dot{q} \in \mathcal{L}_\infty$ with $\dot{r} \in \mathcal{L}_\infty$. Also, $\varphi_s(t) \leq V_s(t) \leq V_s(0)\ \forall t \geq 0$, thus, $\partial\varphi_s/\partial s \in \mathcal{L}_\infty$. Then, from (17) with the assumptions made above and the fact that $Q(r,\dot{q})$ is linear w.r.t. $\partial M_{ij}/\partial q_k$ and $\dot{q}$, we have $\dot{\nu}_s \in \mathcal{L}_\infty$ (and also $\ddot{q} \in \mathcal{L}_\infty$ with $\ddot{r} \in \mathcal{L}_\infty$). Then, from Barbalat's lemma, $\nu_s \to 0$. Also, differentiating (17) with the above assumptions, we can also show that $\ddot{\nu}_s \in \mathcal{L}_\infty$. So, again, from Barbalat's lemma, $\dot{\nu}_s \to 0$. Applying $(\nu_s, \dot{\nu}_s) \to 0$ to (17) then completes the proof. ■

A. Passivity-Based Time-Varying Control

Note that Lem. 2 holds even if $r(t)$ keeps changing. Thus, similar to the persistency of excitation for parameter convergence in adaptive control, if we can drive r in such a way that the null space of $\mathcal{D}_s^T(r)$ also keeps "rotating", the condition (19) will then imply $\partial\varphi_s(s)/\partial s \to 0$ (thus, $s \to s_d$ from our construction of φ_s). This leads into the idea of **passivity-based time-varying control**, as formalized in the following theorem, which is a direct consequence of Lem. 2 and of our construction of $\varphi_s(s)$.

Theorem 2 *Consider the ν_s-dynamics* (6) *with u_s* (16) *and $\beta(r) = 0$. Assume the same as in Lem. 2. Suppose further that, controlling* (7)*, we drive $r(t)$ in such a way that $\forall t' \geq 0$, $\exists$ a bounded $\Delta t \geq 0$ s.t.*

$$\mathcal{D}_s^T(r(t))\left[\frac{\partial\varphi_s(s)}{\partial s}\right]^T = 0 \ \ \forall t \in [t', t'+\Delta t) \ \ \textit{iff} \ \ \frac{\partial\varphi_s(s)}{\partial s} = 0$$

Then, $\nu_s \to 0$ and $s \to s_d$.

Note from (7) that we can drive $r(t)$ to track any desired trajectory $r_o(t) \in \Re^m$ by using

$$u_r := Q_{rs}\nu_s + D_r\ddot{r}_o + Q_r\dot{r}_o - k_d(\dot{r} - \dot{r}_o) - k_p(r - r_o) - \delta_r$$

where $Q_{rs}\nu_s$ is the decoupling control, and $k_d, k_p \in \Re^{m\times m}$ are the control gains. Since we can choose any (well-behaved) $r_o(t)$, achieving the condition in Th. 2 is in essence only restricted by the structure of $\mathcal{D}_s^T(r)$. Note also that, here, we can achieve $r \to r_d$ while keeping $||s - s_d||$ small, simply by replacing $r_o(t)$ in the above u_r by the (constant) r_d, once both ν_s and $\varphi_s(s)$ become small enough. This is because, if $||\nu_s(t')|| \leq \epsilon_1$ and $\varphi_s(t') \leq \epsilon_2$ for a certain $t' \geq 0$ and with small $\epsilon_1, \epsilon_2 > 0$, from (18) with the b_s-dissipation and the fact that $\kappa_s \geq 0$, we will have $\varphi_s(t) \leq \epsilon_1 + \epsilon_2$ for all $t \geq t'$, *regardless* of the motion of r. This will then imply the practical stabilization [1] of s with $r \to r_d$.

Similar varying control was proposed in [6], [7] for general nonholonomic mechanical systems, and our time-varying control here is in fact an extension of that to fully utilizing the system's symmetry and the full control on $\mathcal{R}$.

B. Passivity-Based Switching Control

Suppose that we stabilize r to a certain constant $r_\sigma \in \mathcal{R}$, where $\sigma \in N$ is to embed switching sequence as below. This can be done by using the following simple PD-control:

$$u_r := Q_{rs}\nu_s - k_d\dot{r} - k_p(r - r_\sigma) \tag{20}$$

where k_p, k_d are the control gains for the (unconstrained) r-dynamics (7) on $\mathcal{R}$ with Lagrangian-like structure and passivity. Then, from (19), s will eventually converge to a (switching) manifold $\mathcal{M}_\sigma \subset \mathcal{S}$ as defined by

$$\mathcal{M}_\sigma := \{s \in \mathcal{S} \mid \mathcal{D}_s^T(r_\sigma)\left[\frac{\partial\varphi_s(s)}{\partial s}\right]^T = 0\} \tag{21}$$

Now, suppose that, once s is stabilized to $\mathcal{M}_1$ with $\sigma = 1$ and small ν_s, we trigger the switching s.t. $\sigma \Leftarrow 2$ with $\mathcal{M}_1 \neq \mathcal{M}_2$. Let us denote this switching time by $t_i \geq 0$, and the time when s is stabilized to $\mathcal{M}_2$ with small ν_s by $t_{i+1} > t_i$. At this t_{i+1}, another switching may also be triggered with $\sigma \Leftarrow 1$. Then, from (18), we have the following energetics during the interval $I_i := [t_i, t_{i+1})$:

$$\begin{aligned}\varphi_s(t_{i+1}) - \varphi_s(t_i) &= \kappa_s(t_i) - \kappa_s(t_{i+1}) - \int_{t_i}^{t_{i+1}} ||\nu_s||_{b_s} dt \\ &\leq \kappa_s(t_i) - \int_{t_i}^{t_{i+1}} ||\nu_s||_{b_s}^2 dt\end{aligned} \tag{22}$$

with $\kappa_s(t_{i+1}) \geq 0$. Thus, if the last line of (22) is strictly negative, we will have strict decrease of φ_s, thus, by repeating these switchings, we will eventually have $\varphi_s(t) \to 0$ with $s \to s_d$. Note that this requires the b_s-dissipation absorbs all $\kappa_s(t_i)$. As shown in the below lemma, which extends [8,

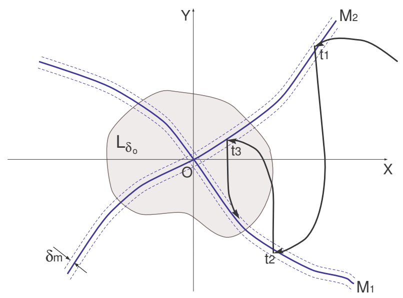

Fig. 2. Illustration of passivity-based switching on $\Re^2$, with two switching manifolds $\mathcal{M}_1$, $\mathcal{M}_2$, their strips with δ_m thickness, and the level set $\mathcal{L}_{\delta_o}$.

Lem.1] from $\Re^2$ to $\mathcal{S} \approx \Re^{n-m}$, this turns out to be indeed possible if: 1) $\kappa_s(t_i)$ is small enough; and 2) the distance D the system traverse on $\mathcal{S}$ between t_i and t_{i+1} is long enough.

Lemma 3 *Given $D > 0$, if $||s(t_{i+1}) - s(t_i)|| \geq D$, there always exists a small $\epsilon_v(D) > 0$ s.t. if $||\nu_s(t_i)|| \leq \epsilon_v$,*

$$\int_{t_i}^{t_{i+1}} ||\nu_s||^2_{b_v} dt > \kappa_s(t_i)$$

where $|| \star ||^2 := \star^T \star$.

Proof: If $\nu_s(t_i) = 0$, with $D > 0$, this claim is trivially hold. Assume $\nu_s(t_i) \neq 0$. Then, the following is true:

$$\int_{I_i} ||\nu_s||^2_{b_s} dt \geq \underline{\sigma}[b_s] \int_{I_i} ||\nu_s||^2 dt = \underline{\sigma}[b_s] \int_{I_i} ||\dot{s}||^2 dt$$
$$\geq \underline{\sigma}[b_s] ||\dot{s}(t_i)||^2 \frac{D}{||\dot{s}(t_i)||} = \underline{\sigma}[b_s] ||\dot{s}(t_i)|| D$$

where $\underline{\sigma}[\star]$ (or $\bar{\sigma}[\star]$, resp.) is the minimum (or maximum, resp.) singular value of $\star$, and $||\nu_s||^2 = ||\dot{s}||^2$ with $\mathcal{D}_s^T \mathcal{D}_s = I$. Also, the second line is because, from the standard result of calculus of variation [17], among the trajectories on $\mathcal{S}$ connecting $s(t_i)$ and $s(t_{i+1})$, the minimizer of $\int_{I_i} ||\dot{s}||^2 dt$ is given by the Lagrange equation with the Lagrangian given by $||\dot{s}||^2$, that is, $\ddot{s} = 0$. This means $\int_{I_i} ||\dot{s}||^2 dt \geq ||\dot{s}(t_i)||^2 (t_{i+1} - t_i)$ with $t_{i+1} - t_i = D/||\dot{s}(t_i)||$. Moreover, since $\kappa_s(t_i) \leq \bar{\sigma}[D_s] ||\dot{s}(t_i)||^2/2$, if we choose $\epsilon_v < 2\underline{\sigma}[b_s] D / \bar{\sigma}[D_s]$ s.t. $||\nu_s|| \leq \epsilon_v$, $\int_{I_i} ||\nu_s||^2_{b_s} dt - \kappa_s(t_i)$ $\geq \underline{\sigma}[b_s] ||\dot{s}(t_i)|| D - \bar{\sigma}[D_s] ||\dot{s}(t_i)||^2/2 > 0$. ∎

This Lem. 3 means that, if we trigger the switching at t_i with $||\nu_s(t_i)|| \leq \epsilon_v$ and this switching requires the system to traverse at least the distance D on $\mathcal{S}$ between the two switching manifolds, we will always have strict decrease of $\varphi_s(s)$, thereby, eventually achieve $s \to s_d$. This is the main idea of **passivity-based switching control** as stated in the following theorem. Of course, we are by no means bound to use just two switching manifolds. Thus, assume p switching manifolds $\mathcal{M}_{\sigma(i)}$ with $\sigma(i) \in \{1, 2, ..., p\}$, where $\sigma(i)$ defines the switching index for $I_{i-1} = [t_{i-1}, t_i)$ ($i = 1, 2, ...$), which is constant during I_{i-1}. Let us also define the "strip" $\bar{\mathcal{M}}_{\sigma(i)}$ of $\mathcal{M}_{\sigma(i)}$ with the thickness of $\delta_m > 0$ s.t. $\bar{\mathcal{M}}_{\sigma(i)} := \{s \in \mathcal{S} \mid \mathrm{dist}(s, \mathcal{M}_{\sigma(i)}) \leq \delta_m\}$, where $\mathrm{dist}(x, y)$ is the minimum Euclidean distance between x and y. See Fig. 2.

Theorem 3 *Suppose that, given $\delta_o > 0$, we can find $\delta_m > 0$ and $D > 0$ s.t., for any $j \in \{1, 2, .., p\}$ and $s \in \mathcal{S}$, if $s \in \bar{\mathcal{M}}_j$, yet, $s \notin \mathcal{L}_{\delta_o}$ (see (15)), $\exists$ a non-empty set $W(s, j) \subset \{1, 2, ...p\}$ s.t.*

$$\mathrm{dist}(s, \bar{\mathcal{M}}_k) \geq D, \quad \forall k \in W(s, j) \tag{23}$$

where $\bar{\mathcal{M}}_\star$ is the strip of $\mathcal{M}_\star$ with δ_m-thickness. Trigger the switching at $t =: t_i > t_{i-1}$ with $\sigma(i+1) \in W(s(t), \sigma(i))$, if 1) $s(t) \notin \mathcal{L}_{\delta_o}$; 2) $\mathrm{dist}(s(t), \mathcal{M}_{\sigma(i)}) \leq \delta_m$; and 3) $||\nu_s(t)|| \leq \epsilon_v(D)$, where $\epsilon_v(D)$ is defined in Lem. 3. Then, $\lim_{t \to \infty} s(t) \in \mathcal{L}_{\delta_o}$.

Proof: Due to Lem. 2, there always exists a time when the switching conditions 2)-3) are satisfied. Suppose that, at such a switching instance t_i, $i \in \{1, 2, ..\}$, $s(t_i) \in \bar{\mathcal{M}}_{\sigma(i)}$, yet, $s(t_i) \notin \mathcal{L}_{\delta_o}$. Then, from (23) with Lem. 3, $\varphi_s(t_{i+1}) < \varphi_s(t_i)$. If still $s(t_{i+1}) \notin \mathcal{L}_{\delta_o}$ at the next switching instance t_{i+1}, we will have $\varphi_s(t_{i+2}) < \varphi_s(t_{i+1})$. By continuing this process, we will have strictly decreasing sequence of $\varphi_s(t_i)$, thus, eventually, will achieve $\varphi_s(t_j) \leq \delta_o$ with $s(t_j) \in \mathcal{L}_{\delta_o} \cap \bar{\mathcal{M}}_{\sigma(j)}$ for a $j \geq i + 2$. ∎

We can also achieve $r \to r_d$ simply by triggering another switching with $r_\sigma \Leftarrow r_d$ in (20), once s is stabilized into $\mathcal{L}_{\delta_o}$ with small $||\nu_s||$. Let us denote this (last) switching time by $\bar{t}$. Then, we have, from (22),

$$\varphi_s(t) \leq \varphi_s(\bar{t}) + \kappa_s(\bar{t}) \leq \delta_o + \frac{1}{2} \bar{\sigma}[D_s] ||\nu_s(\bar{t})||^2$$

for all $t \geq \bar{t}$. That is, if $\nu_s(\bar{t})$ is small enough (e.g. ϵ_v), even with this last switching with $r_\sigma \Leftarrow r_d$, $\varphi_s(s)$ will still be small and so will be $||s - s_d||$ (i.e. practical stabilization), while $r \to r_d$.

Note that the switching conditions 2) and 3) in Th. 3 can be achieved simply by waiting enough for the system to stabilize into $\mathcal{M}_{\sigma(i)}$ with small velocity - see Lem. 2. Also, notice that the decoupling of $Q_{sr}\nu_s, Q_{rs}\dot{r}$ in (16) and (20) are crucial here, since, without them, the energy jumps in the spring k_p of (20) due to the switching of r_σ may flow back into the ν_s-dynamics, thereby, jeopardize the strict decrease of φ_s between two switchings.

Our switching strategy here is state-dependent (e.g. $W(s, j)$ for (23)). In fact, if all the switching manifolds $\mathcal{M}_\star$ are separated from each other outside of $\mathcal{L}_{\delta_o}$, we can make our switching state-invariant (i.e. $W(j)$ instead of $W(s, j)$ for (23)); and also can show that $\mathcal{L}_{\delta_o}$ is an invariants set w.r.t. the switching, including those occurred within $\mathcal{L}_{\delta_o}$ (e.g. [8]). This separation of $\mathcal{M}_\star$, however, is in general not possible if $\dim(\mathcal{S}) \geq 3$: e.g. if $\mathcal{M}_1, \mathcal{M}_2$ are two planes in $\mathcal{S} \approx \Re^3$, their distance is zero even outside of $\mathcal{L}_{\delta_o}$, with their intersection given by a line.

We perform simulations for four unicycle-type non-uniform WMRs, with $\mathcal{R} \approx S^4$ being the space of their orientation θ_i, while $\mathcal{S} \approx \Re^8$ being their Cartesian motion $p_i := (x_i, y_i)$. Then, we can show $\beta(r) = 0$, although $M_2(r) \neq 0$. Then, $\mathcal{R}$ is constraint-free, while $\mathcal{S}$ is not. On $\mathcal{S}$, we define $\varphi_s := k_s \sum_{i=1}^{3} ||p_i - p_{i+1} - p_i^d||^2/2$ to achieve a square formation among them. See Fig. 3 and

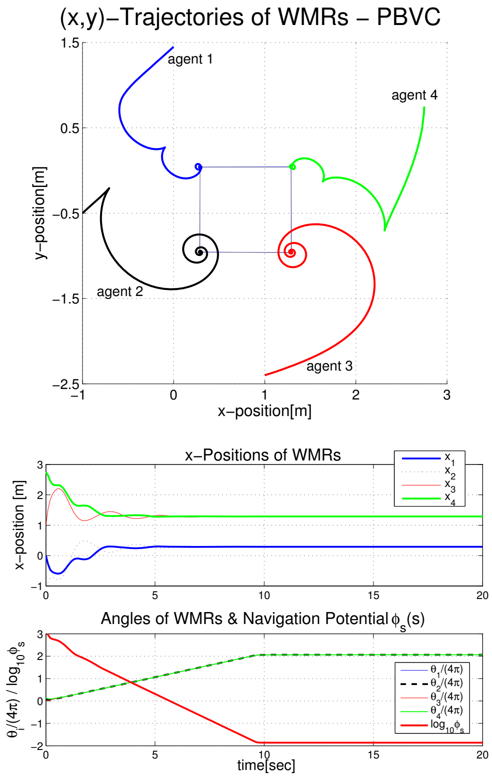

Fig. 3. Passivity-based time-varying stabilization control.

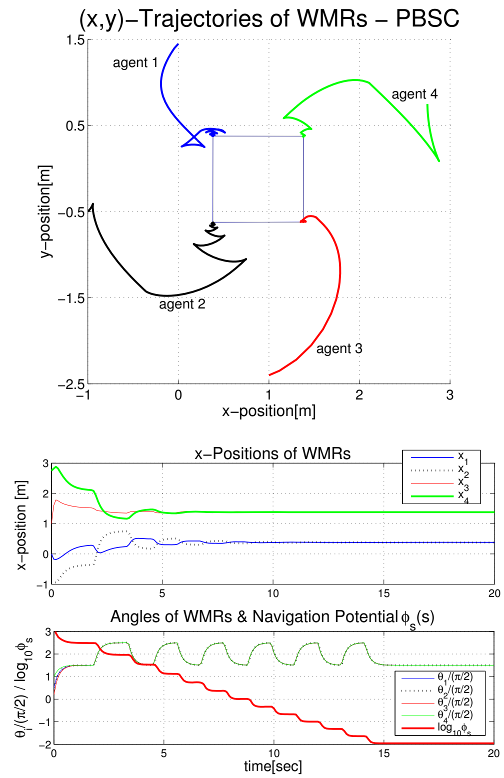

Fig. 4. Passivity-based switching stabilization control.

Fig. 4 for the results with the passivity-based varying and switching controls, respectively, with $\log_{10} \varphi_s$ also shown in their bottom plots. We also choose $\varphi_s \leq 0.02$ as criteria for practical stabilization on $\mathcal{S}$. We can then see that: 1) desired formation is achieved; 2) switching control shows slower convergence due to "stoppings"; 3) although the formation shape is specified, their center point is not; and 4) motion planning is not required since φ_s integrates planning and control. Details on these simulations are omitted here due to page limit, and will be reported in a future publication.

V. Conclusion

In this paper, we consider a certain type of nonholonomic mechanical systems with symmetry. We propose passive configuration decomposition, that enables us to decompose the dynamics and energetics of the Lagrange-D'Alembert dynamics into that on $\mathcal{S}$ and $\mathcal{R}$. We also propose passivity-based varying and switching control laws to achieve practical stabilization, by controlling the decoupled systems on $\mathcal{S}$ and $\mathcal{R}$ individually. Simulation results on multiple WMRs formation stabilization are also shown. Some future research directions include: 1) switching of potential field φ_s on $\mathcal{S}$; 2) equivalence of the conditions in Th. 2 and Th. 3 and their relation to controllability on $\mathcal{S}$; and 3) extension to more general class of nonholonomic systems.

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